Towards Principled Methods for Training Generative Adversarial Networks

Martin Arjovsky & Léon Bottou

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- Close how?

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- Assumptions: continuous with full support.
- Problems: restricted capacity distributes mass.
  Modeling low dimensional distributions is impossible.

Kullback-Leibler Divergence

- Closeness measured by KL divergence (equivalent to ML):

$$\min_{\theta \in \mathbb{R}^d} KL(\mathbb{P}_r \| \mathbb{P}_\theta) = \int_{\mathcal{X}} P_r(x) \log \frac{P_r(x)}{P_\theta(x)} \, \mathrm{d}x$$

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- When $P_r(x) > 0, P_\theta(x) \to 0$ integrand goes to infinity: high cost for mode dropping.
- When $P_\theta(x) > 0, P_r(x) \to 0$ integrand goes to 0: low cost for fake looking samples.

Generative Adversarial Networks (Goodfellow et al.)

- Let $\mathbb{P}_\theta$ be the dist of $g_\theta(Z)$ for some simple (e.g. Gaussian) r.v Z, passed through a complex function.

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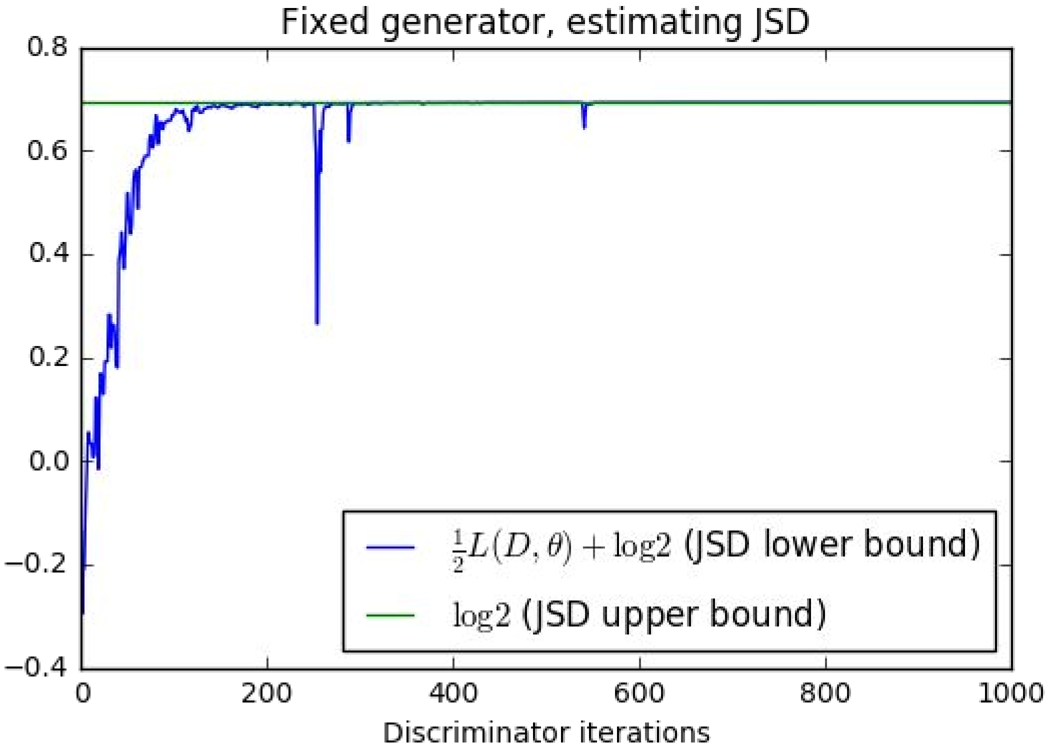
$$JSD(\mathbb{P}_r \| \mathbb{P}_\theta) = \max_D \frac{1}{2} L(D,\theta) + \log 2$$

JSD seems maxed out..



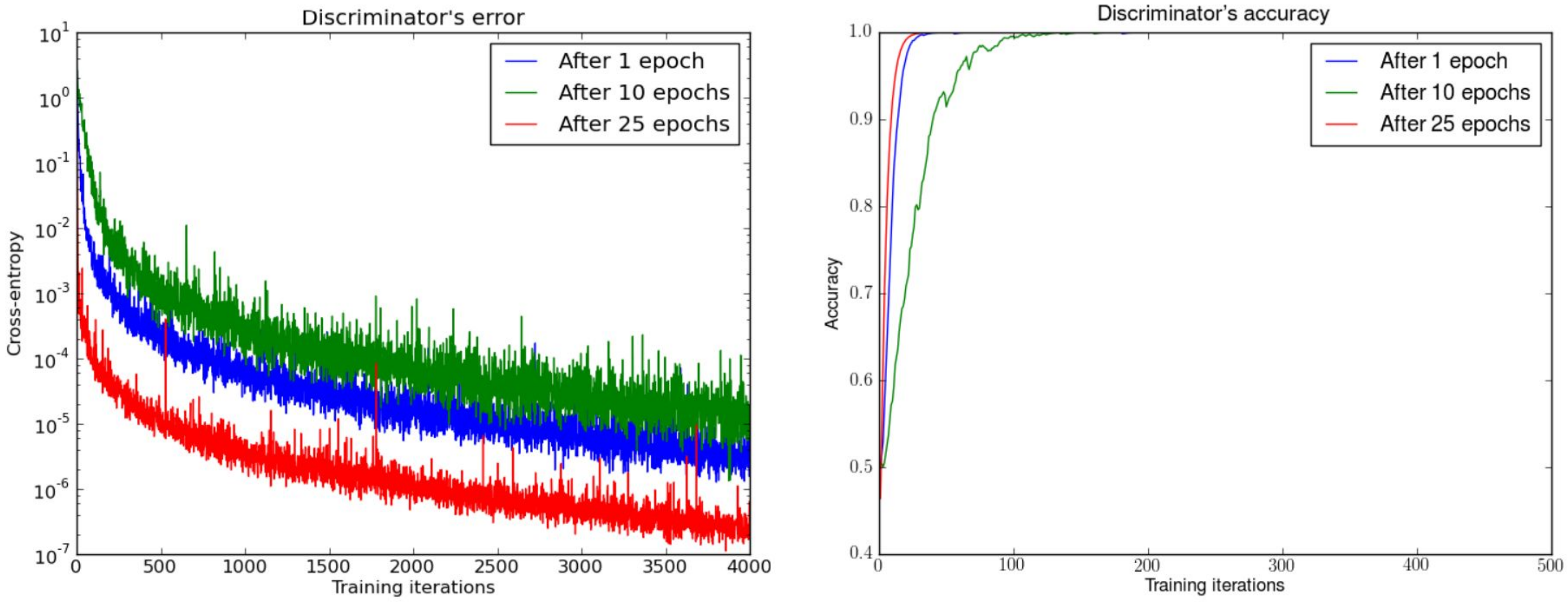
Generative Adversarial Networks

- Under optimal discriminator, minimizes

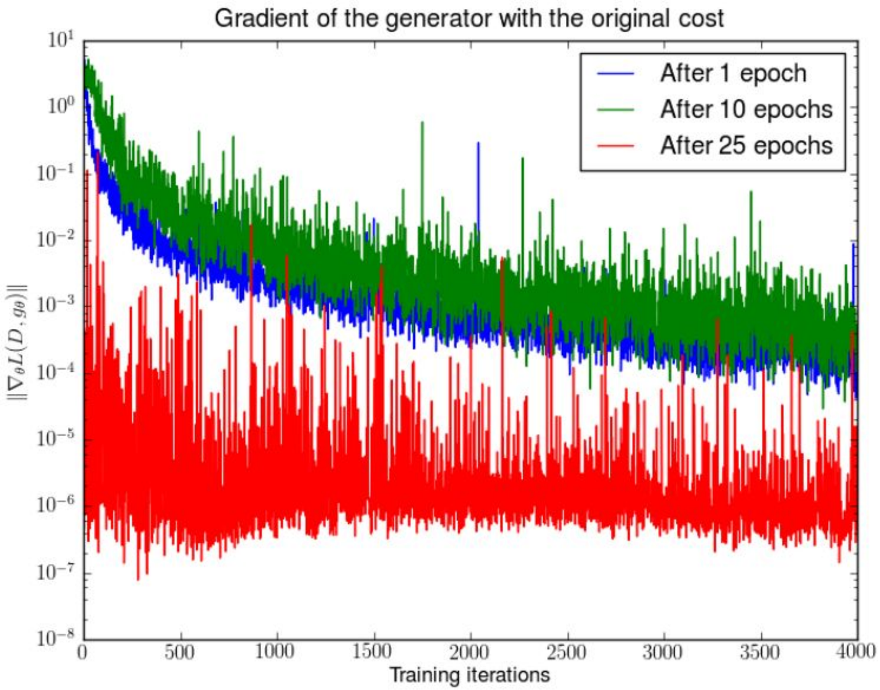
$$\min_{\theta \in \mathbb{R}^d} JSD(\mathbb{P}_r || \mathbb{P}_\theta) = KL(\mathbb{P}_r || \mathbb{P}_m) + KL(\mathbb{P}_\theta || \mathbb{P}_m)$$

- Problems: vanishing gradients very quickly when D’s accuracy is high.

Discriminator is pretty good...



Vanishing gradients, original cost



Alternate update

- Alternate update that has less vanishing gradients

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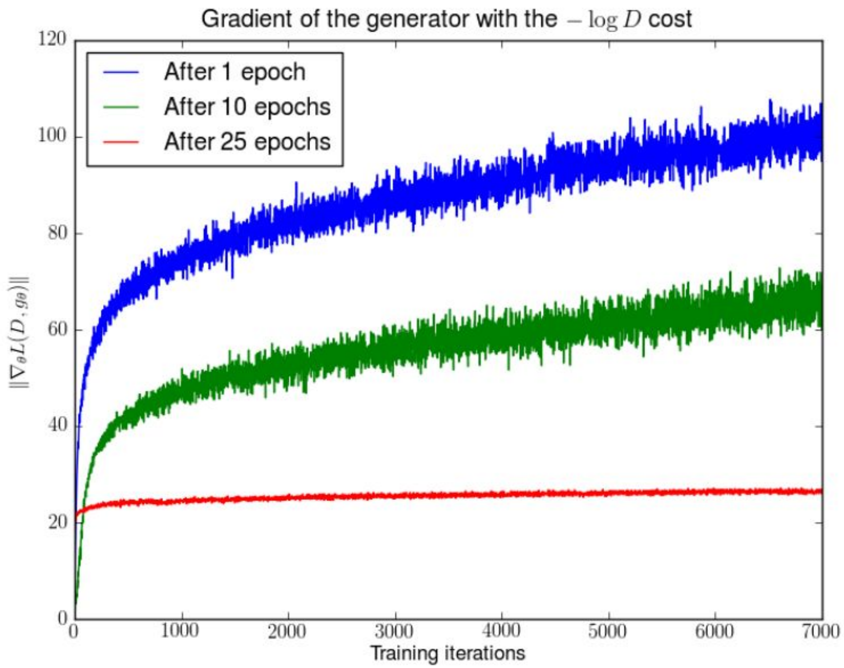
- Problems: JSD with the wrong sign, reverse KL has high mode dropping. Still unstable when D is good.

High variance updates



Problems of GANs (and divergences)

- When $\mathbb{P}_r$ and $\mathbb{P}_\theta$ lie on low dimensional manifolds, there's always a perfect discriminator, that provides no usable gradients.

Manifold picture

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- When $\mathbb{P}_r$ and $\mathbb{P}_\theta$ lie on low dimensional manifolds, there's always a perfect discriminator, that provides no usable gradients.

Theorem 2.2. *Let $\mathbb{P}_r$ and $\mathbb{P}_g$ be two distributions that have support contained in two closed manifolds $\mathcal{M}$ and $\mathcal{P}$ that don't perfectly align and don't have full dimension. We further assume that $\mathbb{P}_r$ and $\mathbb{P}_g$ are continuous in their respective manifolds, meaning that if there is a set A with measure 0 in $\mathcal{M}$, then $\mathbb{P}_r(A) = 0$ (and analogously for $\mathbb{P}_g$). Then, there exists an optimal discriminator $D^* : \mathcal{X} \to [0, 1]$ that has accuracy 1 and for almost any x in $\mathcal{M}$ or $\mathcal{P}$, D^* is smooth in a neighbourhood of x and $\nabla_x D^*(x) = 0$.*

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- Under the same assumptions

$$JSD(\mathbb{P}_r||\mathbb{P}_\theta) = \log 2$$
$$KL(\mathbb{P}_r||\mathbb{P}_\theta) = +\infty$$
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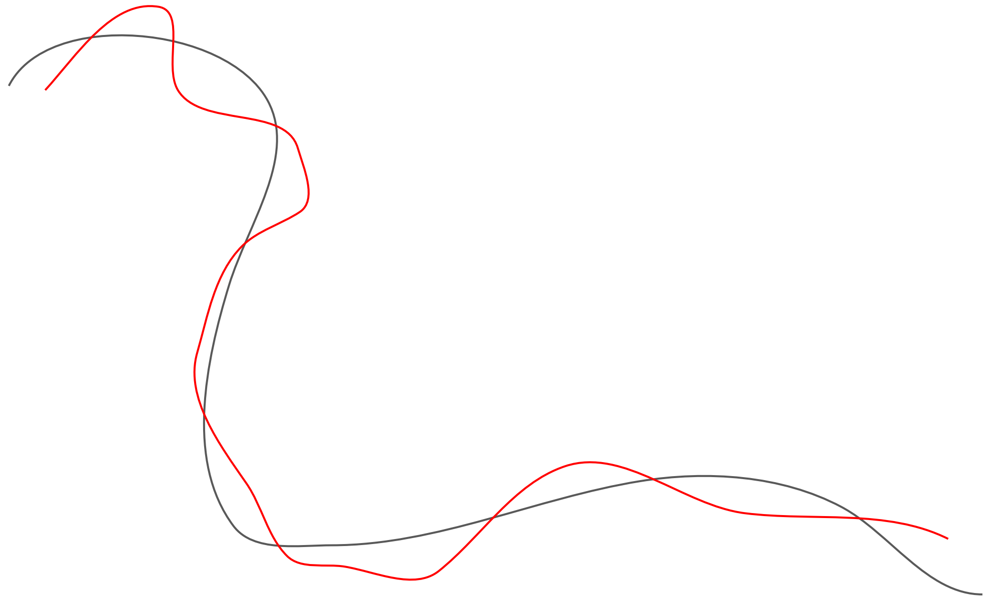
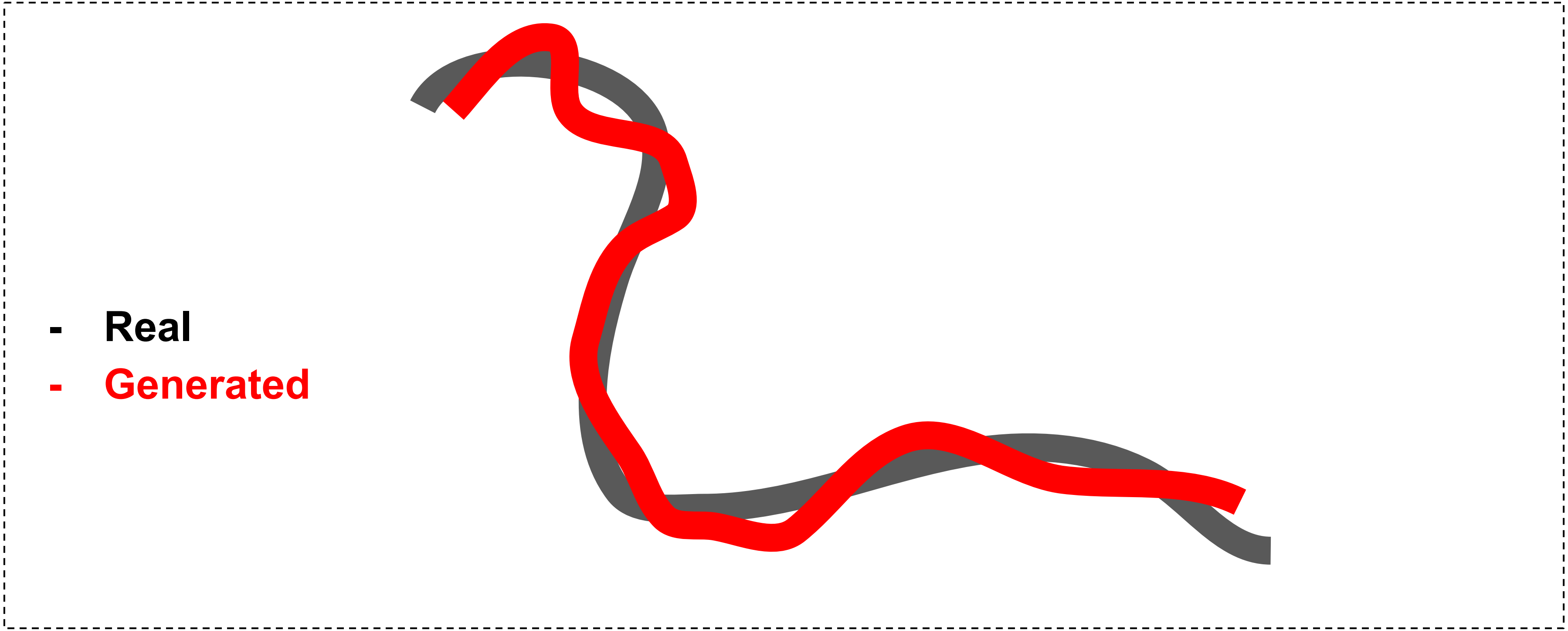
- Matching noisy distributions amounts to matching the underlying ones.

Manifold picture



- Real
- Generated

Manifold picture with noise



A first step to a solution

Theorem 3.2. *Let $\mathbb{P}_r$ and $\mathbb{P}_g$ be two distributions with support on $\mathcal{M}$ and $\mathcal{P}$ respectively, with $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Then, the gradient passed to the generator has the form*

$$\begin{aligned} &\mathbb{E}_{z\sim p(z)}\left[\nabla_\theta \log(1 - D^*(g_\theta(z)))\right] \qquad (4)\\ &= \mathbb{E}_{z\sim p(z)}\left[a(z)\int_{\mathcal{M}} P_\epsilon(g_\theta(z) - y)\nabla_\theta \|g_\theta(z) - y\|^2 \, \mathrm{d}\mathbb{P}_r(y)\right.\\ &\left. - b(z)\int_{\mathcal{P}} P_\epsilon(g_\theta(z) - y)\nabla_\theta \|g_\theta(z) - y\|^2 \, \mathrm{d}\mathbb{P}_g(y)\right] \end{aligned}$$

We move our samples $g_\theta(z)$ towards point in the data manifold, weighted by their probability and distance to our samples.

Theoretical guarantee

Theorem 3.3. *Let $\mathbb{P}_r$ and $\mathbb{P}_g$ be any two distributions, and ϵ be a random vector with mean 0 and variance V. If $\mathbb{P}_{r+\epsilon}$ and $\mathbb{P}_{g+\epsilon}$ have support contained on a ball of diameter C, then* [6]

$$W(\mathbb{P}_r, \mathbb{P}_g) \leq 2V^{\frac{1}{2}} + 2C\sqrt{JSD(\mathbb{P}_{r+\epsilon}||\mathbb{P}_{g+\epsilon})} \tag{6}$$

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- We can reduce the first summand by annealing the noise, the second one by optimizing with noise.

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- Optimizing an approximation of Wasserstein directly is doable. (Arjovsky, Chintala & Bottou, 2017)
- Different ways to do this. (Gulrajani et al. 2017)
- Time to scale up!


That's all Folks!
SUPERB wallpapers